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Completeness of superintegrability in two-dimensional constant-curvature spaces

E G Kalnins[1], **J M Kress**[1], **G S Pogosyan**[2,4] **and W Miller Jr**[3]

[1] Department of Mathematics, University of Waikato, Hamilton, New Zealand
[2] Centro de Ciencias Fisicas, Universidad Nacional Autonoma de Mexico, Apartado Postal 48–3, 62251 Cuernavaca, Morelos, Mexico
[3] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

E-mail: e.kalnins@waikato.ac.nz, jonathan@math.waikato.ac.nz, pogosyan@thsun1.jinr.dubna.su and miller@ima.umn.edu

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Abstract

We classify the Hamiltonians $H = p_x^2 + p_y^2 + V(x, y)$ of all classical superintegrable systems in two-dimensional complex Euclidean space with two additional second-order constants of the motion. We similarly classify the superintegrable Hamiltonians $H = J_1^2 + J_2^2 + J_3^2 + V(x, y, z)$ on the complex two-sphere where $x^2 + y^2 + z^2 = 1$. This is achieved in all generality using properties of the complex Euclidean group and the complex orthogonal group.

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1. Introduction

It is known from classical mechanics that a mechanical system with n degrees of freedom is completely integrable if there are n functionally independent constants of the motion which are mutually in involution [1]. The idea of a superintegrable system is that there exist more than n functionally independent constants of the motion, but not necessarily in involution. If there are $2n - 1$ such constants the system is said to be *maximally superintegrable* or just *superintegrable* [2–5]. Here we consider only the case where there exist $2n - 1$ functionally independent constants of the motion (including the Hamiltonian) that are quadratic in the momenta. Rañada [5] investigated such systems and noted that many could be found in Drach's list of potentials admitting constants cubic in the momenta[5]. In the papers [6–8] we have given a complete classification of all *non-degenerate potentials* on complex Euclidean 2-space and on the complex two-sphere that give rise to superintegrable systems. (For example in [6] we have calculated all the inequivalent superintegrable potentials V that are *non-degenerate* in the

[4] Permanent address: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region 14980, Russia.
[5] These are **(E1, E2, E7, E9, E16, E19, E20)** in our notation.

sense that they depend uniquely on four arbitrary parameters, i.e. one can prescribe the values of V, V_x, V_y, V_{yy} arbitrarily at any regular point (x_0, y_0) and these values determine $V(x, y)$ uniquely.) In this paper we relax this requirement and ask the same question but without the condition of non-degeneracy: for which potentials in two dimensions do there exist at least two constants of the motion

$$A_j = a_j(x, y)p_x^2 + b_j(x, y)p_y^2 + c_j(x, y)p_x p_y + d_j(x, y) = A'_j + d_j \qquad j = 1, 2 \tag{1}$$

in addition to the Euclidean space Hamiltonian

$$H = p_x^2 + p_y^2 + V(x, y) \tag{2}$$

i.e. $\{H, A_j\} = 0,\ j = 1, 2$, and such that the $2n - 1 = 3$ constants of the motion $H = A_0, A_1, A_2$ are functionally independent on phase space? We will do the same for the Hamiltonian on the complex two-sphere

$$H = J_1^2 + J_2^2 + J_3^2 + V(x, y, z) \tag{3}$$

where $x^2 + y^2 + z^2 = 1$ and $J_1 = yp_z - zp_y,\ J_2 = zp_x - xp_z,\ J_3 = xp_y - yp_x$. We give a complete solution. The computations are lengthy, and throughout we have made use of a computer algebra package. We give many details in the first few examples, to make our method clear.

In [6–8] we worked out the structure of the quadratic algebra for each of the non-degenerate potentials. In this paper we supplement those results by computing the quadratic algebras for the non-constant degenerate potentials. Also we correct a few errors and fill in some gaps in those earlier papers.

2. Superintegrability in $E_{2,C}$

In the computations to follow, quite often we will be considering systems such that the corresponding Hamilton Jacobi equation $H = E$, (2), can be solved by the method of separation of variables. Then $p_x = \frac{\partial S}{\partial x},\ p_y = \frac{\partial S}{\partial y}$ and there is a complete integral of the form

$$S = U(u, E, \lambda) + V(v, E, \lambda)$$

for separable coordinates $u = u(x, y),\ v = v(x, y)$ and some separation constant λ. This will not always be the case, but when separation is possible a knowledge of the separable coordinates u, v will greatly simplify our computations. In general we will use the structure of the complex Euclidean group $E(2, C)$ and its Lie algebra $e(2, C)$ to solve our problem. The elements $L_1 = p_x,\ L_2 = p_y$ and $L_3 = M = xp_y - yp_x$ form a basis for $e(2, C)$ under the Poisson bracket, and quadratic elements $L_i L_j,\quad i \leqslant j$ form a basis for all purely quadratic functions A' such that $\{A', p_x^2 + p_y^2\} = 0$. Thus the quadratic integrals A_j can be written in the form

$$A_j = a_j^{k\ell} L_k L_\ell + d_j(x, y) = A'_j + d_j \tag{4}$$

for suitable constants $a_j^{k\ell} = a_j^{\ell k}$.

We will not regard Euclidean space Hamiltonians as essentially different if they are related by a Euclidean transformation. Because of this we can use Euclidean group transformations to simplify the expressions for the A_j and classify them into equivalence classes. If we do this then there are equivalence classes of constants whose typical representatives are [9]

$$\begin{array}{llll} p_x^2 & (p_x + \mathrm{i}p_y)^2 & M^2 & M(p_x + \mathrm{i}p_y) + (p_x - \mathrm{i}p_y)^2 \\ M^2 + (p_x + \mathrm{i}p_y)^2 & Mp_x & M^2 + c^2 p_x^2 & M(p_x + \mathrm{i}p_y). \end{array} \tag{5}$$

(Note that, up to the addition of an arbitrary multiple of the Casimir element $p_x^2 + p_y^2$, this is simply a choice of a representative on each distinct orbit of second-order elements in the enveloping algebra of $e(2, C)$ under the adjoint action of $E(2, C)$.) Without loss of generality, we can assume that A_1' coincides with one of these representatives and use the defining relations $\{H, A_j\} = 0$ for constants of the motion, to determine the general form of the second constant. This is a particularly useful strategy since all but the last of the list (5) of representatives has a form that implies separation of variables in at least one coordinate system. If we use this fact then we see that the corresponding potential V must have the form implied by separation, and the requirement of an extra constant of the motion implies strong conditions on this functional form. This will greatly simplify our computations. For all but one potential, we find that the associated constants determine more than one separating coordinate system. These are listed in appendix A.

We now deal with each of these cases individually. Consider the first constant in our complete family of equivalence classes. We assume our Hamiltonian has a constant of the form

$$A_1 = p_x^2 + d_1(x, y). \tag{6}$$

The condition $\{H, A_1\} = 0$ implies $\partial_y d_1(x, y) = 0$, $\partial_x(V - d_1(x, y)) = 0$, so we can assume that the Hamiltonian has the form

$$H = A_0 = p_x^2 + p_y^2 + f(x) + h(y) \tag{7}$$

where $d_1(x, y) = f(x)$. For superintegrability we must have one additional constant of the motion, which can be written

$$A_2 = a_2^{k\ell} L_k L_\ell + d_2(x, y). \tag{8}$$

Since we can always add linear combinations of A_0 and A_1 to A_2 without changing the system, we can assume that

$$A_2 = AM^2 + BMp_x + CMp_y + Dp_x p_y + d_2(x, y) \tag{9}$$

where A, B, C and D are constants, not all zero.

Because of the forms of A_0 and A_1 we can always apply translations to A_2 in order to simplify its form. Suppose $A \neq 0$ in (9). Then, normalizing so that $A = 1$, by appropriate translations in x and y we can pass to a new Cartesian coordinate system in which $B = C = 0$. The condition $\{A_0, A_2\} = 0$ then determines the possible forms of A_2. Indeed, equating coefficients of p_x and p_y, we find

$$\partial_y d_2(x, y) = (D - 2xy)f'(x) + 2x^2 h'(y) \qquad \partial_x d_2(x, y) = 2y^2 f'(x) + (D - 2xy)h'(y).$$

Equating the cross partial derivatives of $d_2(x, y)$ we obtain the condition

$$\left(f'' + \frac{3}{x}f'\right) - \left(h'' + \frac{3}{y}h'\right) = \frac{D}{2xy}(h'' - f''). \tag{10}$$

If $D = 0$, the variables separate and we find a well known non-degenerate superintegrable potential:

(E1) $V = \omega^2(x^2 + y^2) + \dfrac{\alpha}{x^2} + \dfrac{\beta}{y^2}$.

The additional constant has the form

$$A_2 = M^2 + \alpha\frac{y^2}{x^2} + \beta\frac{x^2}{y^2}.$$

This example together with its Poisson bracket relations is well studied [6, 10].

If $A = 0$ but $B^2 + C^2 \neq 0$ in (9) then we can rotate coordinates so that $B = 0$, normalize so that $C = 1$, translate to obtain $D = 0$ and find the non-degenerate potential:

(E2) $V = \omega^2(4x^2 + y^2) + \alpha x + \frac{\beta}{y^2}$.
The additional constant has the form

$$A_2 = Mp_y + d_2(x, y).$$

There are no further non-degenerate potentials separating in Cartesian coordinates. We now return to the case $A \neq 0$ and suppose that $D \neq 0$. Then from (10) we see that h'' and f'' must satisfy a functional equation of the form $h''(y) - f''(x) = xy(G(y) + F(x))$ for some functions G and F. Solving this equation and substituting back into (10), we find that the variables separate and we obtain the solution:

(E3) $V = \omega^2(x^2 + y^2)$.
In this case, $d_1(x, y) = \omega^2 x^2$, and there are two additional constants, one of which is first order. They can be taken in the form

$$A_2 = p_x p_y + \omega^2 xy \qquad X = M.$$

The Poisson bracket relations for these constants are

$$\{A_1, X\} = 2A_2 \qquad \{A_2, X\} = A_0 - 2A_1 \qquad \{A_1, A_2\} = -2\omega^2 X.$$

Since A_0, A_1 and A_2 are functionally independent, all constants of the motion are functions of these. It is easy to verify that X satisfies the functional relation

$$A_2^2 - A_1(A_0 - A_1) + \omega^2 X^2 = 0.$$

This example is the harmonic oscillator in two dimensions.
If $A = 0$ and $B^2 + C^2 = 0$ but $B \neq 0$, we can take $B = 1$ and $C = i$ (by mapping y to $-y$ if necessary) and translate to obtain $D = 0$. A straightforward computation gives:

(E4) $V = \alpha(x + \mathrm{i}y)$.
Here $d_1(x, y) = \alpha x$ and the Hamiltonian admits two extra constants, one of which is first order,

$$A_2 = M(p_x + \mathrm{i}p_y) + \frac{\mathrm{i}\alpha}{4}(x + \mathrm{i}y)^2 \qquad X = p_x + \mathrm{i}p_y.$$

The Poisson bracket relations take the form

$$\{A_1, X\} = -\alpha \qquad \{A_2, X\} = \mathrm{i}X^2 \qquad \{A_1, A_2\} = -\mathrm{i}X^3 + 2\mathrm{i}A_1 X - \mathrm{i}A_0 X$$

with the functional relation

$$A_0^2 + X^2(2A_0 - 4A_1 + X^2) + 4\mathrm{i}\alpha A_2 = 0.$$

If $A = B = C = 0$ then we normalize $D = 2$ and find the non-degenerate potential $V = \omega^2(x^2 + y^2) + \alpha x + \beta y$. For a fixed choice of the parameters, the Hamiltonian admits the first-order constant of the motion

$$X = 2\omega^2 M + \alpha p_y - \beta p_x.$$

By an appropriate translation we can obtain $X' = M$, which is case **(E3)**.

There are special cases of potentials **(E1)** and **(E2)** such that the Hamiltonian admits more than two constants of the motion. The possibilities are:

(E5) $V = \alpha x$.

Since, in this case, $A_0 - A_1 = p_y^2$, we can replace A_1 with the first-order constant p_y and take the additional constants as

$$A_2 = M p_y - \frac{\alpha}{4} y^2 \qquad A_3 = p_x p_y + \frac{\alpha}{2} y \qquad X = p_y.$$

They satisfy the Poisson bracket relations

$$\{A_2, X\} = A_3 \qquad \{A_3, X\} = -\frac{\alpha}{2} \qquad \{A_3, A_2\} = 2X^3 - A_0 X$$

and the functional relation

$$A_3^2 + X^4 - A_0 X^2 + \alpha A_2 = 0.$$

(E6) $V = \frac{\alpha}{x^2}$.

As for the previous case, we can replace A_1 with p_y. The additional constants are

$$A_2 = M p_x - \frac{\alpha y}{x^2} \qquad A_3 = M^2 + \frac{\alpha y^2}{x^2} \qquad X = p_y.$$

Their Poisson brackets are

$$\{A_2, X\} = A_0 - X^2 \qquad \{A_3, X\} = 2A_2 \qquad \{A_3, A_2\} = -2XA_3 - 2\alpha X$$

and they satisfy the functional relation

$$A_2^2 - A_3(A_0 - X^2) + \alpha X^2 = 0.$$

This concludes the list of possible potentials corresponding to the first equivalence class of second-order elements in the enveloping algebra of $e(2, C)$.

For orbits of the second type, the constant of the motion A_1 has the form

$$A_1 = p_-^2 + d_1(x, y). \qquad (11)$$

(We adopt the notation $p_\pm = p_x \pm \mathrm{i} p_y$, $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$.) It follows from the relation $\{A_0, A_1\} = 0$ that for a Hamiltonian to admit a constant of the motion of the form (11) the potential V must have the form

$$V = f(\bar{z})z + h(\bar{z})$$

for some functions f and h. We can assume

$$A_2 = AM^2 + BMp_+ + CMp_- + Dp_+^2 + d_2(z, \bar{z}).$$

There are several possibilities. In the first case we assume $A \neq 0$. Then we can normalize $A = 1$, translate to obtain $B = C = 0$ and write $D = c^2/2$. We find the non-degenerate potential [6]:

(E7) $V = \frac{\alpha \bar{z}}{\sqrt{\bar{z}^2 - c^2}} + \frac{\beta z}{\sqrt{\bar{z}^2 - c^2}(\bar{z} + \sqrt{\bar{z}^2 - c^2})^2} + \gamma z\bar{z}$.

Here the second constant of the motion can be taken in the form

$$A_2 = M^2 + c^2 p_x^2 + d_2(x, y).$$

The limiting case of this as $c \to 0$ gives the potential [6]:

(E8) $V = \frac{\alpha z}{\bar{z}^3} + \frac{\beta}{\bar{z}^2} + \gamma z\bar{z}$.

Here the second constant of the motion has the form

$$A_2 = M^2 + d_2(x, y).$$

If $A = 0$ but $BC \neq 0$, we can normalize and rotate to obtain $B = -\mathrm{i}/2$ and $C = \mathrm{i}/2$ and translate to obtain $D = 0$. We obtain the non-degenerate potential [6]:

(E9) $V = \frac{\alpha}{\sqrt{\bar z}} + \beta x + \frac{\gamma(x+\bar z)}{\sqrt{\bar z}}$.

The second constant of the motion is

$$A_2 = Mp_y + d_2(z, \bar z).$$

If $A = BC = 0$ but $C \neq 0$ and $D \neq 0$ we normalize $C = 4\mathrm{i}$ and rotate so that $D = 1$ to obtain the non-degenerate potential [6]:

(E10) $V = \alpha \bar z + \beta(z - \frac{3}{2}\bar z^2) + \gamma(z\bar z - \frac{1}{2}\bar z^3)$.

Here the second constant of the motion has the form

$$A_2 = 4\mathrm{i}Mp_- + p_+^2 + d_2(x, y).$$

If $A = BC = 0$ but $B \neq 0$ (or if $C \neq 0$, $B = 0$ and $D = 0$ and we reflect $y \to -y$), we can normalize $B = 1$ and translate so that $D = 0$ to obtain the non-degenerate potential:

(E11) $V = \alpha z + \frac{\beta z}{\sqrt{\bar z}} + \frac{\gamma}{\sqrt{\bar z}}$.

Here the second constant of the motion has the form

$$A_2 = Mp_+ + d_2(x, y).$$

There are special cases of potentials **(E7, E8, E11)** that admit two extra constants of the motion. In each of these cases $A_1 = p_-^2$, i.e. $d_1(x, y) = 0$ and hence $X = p_-$ is a constant of the motion. The possibilities are:

(E12) $V = \frac{\alpha \bar z}{\sqrt{\bar z^2 + c^2}}$

with the constants of motion given by

$$X = p_- \qquad A_2 = M^2 - \frac{c^2}{4}p_+^2 - \frac{\alpha c^2 z}{2\sqrt{\bar z^2 + c^2}} \qquad A_3 = Mp_- + \frac{\mathrm{i}\alpha c^2}{2\sqrt{\bar z^2 + c^2}}.$$

The Poisson bracket relations are

$$\{X, A_2\} = 2\mathrm{i}A_3 \qquad \{X, A_3\} = \mathrm{i}X^2 \qquad \{A_2, A_3\} = -2\mathrm{i}XA_2,$$

with the functional relation

$$A_3^2 - X^2 A_2 - \frac{c^2}{4}A_0^2 + \frac{\alpha^2 c^2}{4} = 0.$$

(E13) $V = \frac{\alpha}{\sqrt{\bar z}}$

with the constants of motion given by

$$X = p_- \qquad A_2 = Mp_+ + \frac{\mathrm{i}\alpha z}{2\sqrt{\bar z}} \qquad A_3 = Mp_- + \frac{\mathrm{i}\alpha}{2}\sqrt{\bar z}.$$

The Poisson bracket relations are

$$\{X, A_2\} = \mathrm{i}A_0 \qquad \{X, A_3\} = \mathrm{i}X^2 \qquad \{A_2, A_3\} = -2\mathrm{i}XA_2,$$

with the functional relation

$$A_3A_0 - X^2A_2 - \frac{\mathrm{i}}{2}\alpha^2 = 0.$$

(E14) $V = \frac{\alpha}{\bar z^2}$

with the constants of the motion given by

$$X = p_- \qquad A_2 = Mp_- - \frac{\mathrm{i}\alpha}{\bar z} \qquad A_3 = M^2 + \frac{\alpha z}{\bar z}.$$

The Poisson bracket relations are

$$\{X, A_2\} = \mathrm{i}X^2 \qquad \{X, A_3\} = 2\mathrm{i}A_2 \qquad \{A_2, A_3\} = 2\mathrm{i}XA_3$$

with the corresponding functional relation

$$A_2^2 - A_3X^2 + \alpha A_0 = 0.$$

(E15) $V = h(\bar{z})$

Here h is any function of $\bar{z}$, not necessarily as already given above. A constant of the motion always exists of the form

$$A_2 = Mp_- + \frac{i}{2}\int \bar{z}\frac{dh}{d\bar{z}}\,d\bar{z}$$

in addition to the constant $X = p_-$. Indeed we might take $h(\bar{z}) = \alpha\bar{z}^2$, in which case

$$A_2 = Mp_- + \frac{i}{3}\alpha\bar{z}^3.$$

This is an example of a potential for which separation of variables occurs in only one coordinate system [8].

For orbits of the third type, the constant of the motion A_1 has the form

$$A_1 = M^2 + d_1(x, y). \tag{12}$$

In this case $V = f(r) + \frac{h(\theta)}{r^2}$ where r and θ are polar coordinates (see appendix A), and $d_1(x, y) = h(\theta)$. We can assume that the second constant takes the general form

$$A_2 = AMp_+ + BMp_- + Cp_+^2 + Dp_-^2 + d_2(x, y). \tag{13}$$

If $AB \neq 0$ then we can rotate and normalize to obtain $A = -B = -i/2$ and translate to achieve $C = D = 0$. This gives us the non-degenerate potential [6]:

(E16) $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \frac{\beta}{x+\sqrt{x^2+y^2}} + \frac{\gamma}{x-\sqrt{x^2+y^2}}\right).$

Here the extra constant of the motion has the form

$$A_2 = Mp_y + d_2(x, y).$$

If $AB = 0$, by letting $y \to -y$ if necessary, we can normalize so that $A = 1$, $B = 0$ and translate to obtain $C = D = 0$. This produces the non-degenerate potential:

(E17) $V = \frac{\alpha}{\sqrt{z\bar{z}}} + \frac{\beta}{z^2} + \frac{\gamma}{z\sqrt{z\bar{z}}}.$

Here the extra constant of the motion has the form

$$A_2 = Mp_+ + d_2(x, y).$$

If $A = B = 0$, the various possibilities have already been included under previous cases.

There is one special case where an extra constant of the motion exists. If A_1 is M^2, so that $d_1(x, y) = 0$ and M is a constant of the motion, then the only additional potential is:

(E18) $V = \frac{\alpha}{\sqrt{x^2+y^2}}.$

The constants of the motion can be taken as

$$A_2 = Mp_x - \frac{\alpha}{2}\frac{y}{\sqrt{x^2+y^2}} \qquad A_3 = Mp_y + \frac{\alpha}{2}\frac{x}{\sqrt{x^2+y^2}} \qquad X = M.$$

The Poisson bracket relations are

$$\{X, A_2\} = -A_3 \qquad \{X, A_3\} = A_2 \qquad \{A_2, A_3\} = XA_0$$

and these constants satisfy

$$A_2^2 + A_3^2 - X^2A_0 - \frac{\alpha^2}{4} = 0.$$

This is the well known Coulomb problem in two dimensions.

For orbits of the fourth type, the constant of the motion A_1 has the form

$$A_1 = Mp_+ + p_-^2 + d_1(x, y) \tag{14}$$

corresponding to semi-hyperbolic coordinates (see appendix A). However, the only superintegrable potentials associated with this constant of the motion have already been considered **(E4, E10, E11, E14)**.

For orbits of the fifth type, the first constant of the motion has the form

$$A_1 = M^2 + p_+^2 + d_1(x, y) \tag{15}$$

corresponding to hyperbolic coordinates (see appendix A). The second constant of the motion can be written in the form

$$A_2 = AMp_+ + BMp_- + Cp_+^2 + Dp_-^2 + d_2(x, y). \tag{16}$$

If $AB \neq 0$ there are no cases with non-constant potential. If $AB = 0$, $|A| + |B| > 0$ there are two new non-degenerate cases:

(E19) $V = \frac{\alpha\bar{z}}{\sqrt{\bar{z}^2-4}} + \frac{\beta}{\sqrt{z(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}-2)}}$
where the additional constant is

$$A_2 = Mp_- + d_2(x, y).$$

The remaining possibilities have been listed earlier.

For orbits of the sixth type, the first constant of the motion has the form

$$A_1 = Mp_x + d_1(x, y) \tag{17}$$

corresponding to parabolic coordinates (see appendix A). There is only one (non-degenerate) case that is not already listed above [6]:

(E20) $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \beta\sqrt{x + \sqrt{x^2+y^2}} + \gamma\sqrt{x - \sqrt{x^2+y^2}}\right)$
where the extra constant of the motion is

$$A_2 = Mp_y + d_2(x, y).$$

Note that although this potential only separates in parabolic coordinate systems, it separates in more than one such coordinate system and hence is multiseparable.

For orbits of the seventh type, the first constant of the motion has the form

$$A_1 = M^2 + c^2 p_x^2 + d_1(x, y) \tag{18}$$

corresponding to elliptic coordinates (see appendix A); however, all superintegrable potentials separating in an elliptic coordinate system have already been listed.

The last orbit on our list of equivalence classes has a typical representative $A_1 = Mp_+ + d_1(x, y)$. The second constant of the motion A_2 must lie on the equivalence class of one of the eight canonical types (5). Therefore, by a Euclidean group motion (including reflections), and by adding multiples of A_0 if necessary, we can assume that the leading terms of A_2 are equal to one of the eight representatives (5). Under this transformation A_1 will be mapped to a constant of motion of the form $\tilde{A}_1 = Mp_\pm + ap_\pm^2 + \tilde{d}_1(x, y)$. For seven of these representatives we have already listed all possible potentials above. Therefore the only new case we need consider is when A_2 transforms to $\tilde{A}_2 = Mp_+ + \tilde{d}_2(x, y)$. Since $\tilde{A}_1$, $\tilde{A}_2$ are functionally independent constants of the motion, we must have either $\tilde{A}_1 = Mp_- + ap_-^2 + d_1(x, y)$ or $a \neq 0$. Consequently the potential under consideration must admit a quadratic constant of the form $p_+^2 + d_3(x, y)$ or one that can be further transformed to $Mp_x + d_3(x, y)$. However, we have already listed all superintegrable potentials admitting a constant of one these forms. Thus there are no new potentials corresponding to this orbit.

This completes the list of possible potentials involved in our problem. As a consequence we see that the list of 20 potentials that we have calculated completely solves the problem in two dimensions of when a potential added to a flat space admits more than one quadratic constant of the motion. All other cases are equivalent to these via proper complex Euclidean transformations and reflections.

3. Superintegrability on the complex two-sphere

We can also solve the similar problem on the complex sphere. Our basic problem is to find the superintegrable potentials for the solution of the Hamilton–Jacobi equation on the complex two-sphere,

$$H = J_1^2 + J_2^2 + J_3^2 + V(x, y, z) = E \tag{19}$$

with x, y and z subject to the constraint $x^2+y^2+z^2 = 1$, and $J_1 = yp_z - zp_y$, $J_2 = zp_x - xp_z$ and $J_3 = xp_y - yp_x$. There are five inequivalent separable coordinate systems for the zero-potential equation (19) and five different quadratic orbits (see appendix B). Typical representatives of these orbit classes are

$$\begin{array}{lll} (J_1 - \mathrm{i}J_2)^2 & J_3(J_1 - \mathrm{i}J_2) & (J_1 + \mathrm{i}J_2) - J_3^2 \\ J_3^2 & J_1^2 + r^2 J_2^2 \quad (r \neq \pm 1, |r| \leqslant 1). & \end{array} \tag{20}$$

We can proceed as we have in the case of the complex Euclidean plane. We consider one of our two quadratic constants to correspond to one of the representatives, (20), hence coming from a separable coordinate system in standard form. The potential must then have an explicit separable form in the appropriate coordinates. We then ask when there exists an additional quadratic constant and what conditions this imposes on our potential. Potentials are considered as equivalent if they are related by an action of the complex orthogonal group O(3), including reflections. For background information about this problem, see [7, 10].

Unlike Euclidean space, all superintegrable potentials are multiseparable. A method for determining the type of separating coordinate from a given constant is described in appendix B and various possibilities for each potential found below are listed.

We consider first those systems that separate in horospherical coordinates. Thus, there is a quadratic constant of the form

$$A_1 = {J_-}^2 + d_1(x, y, z).$$

(We adopt the notation $J_\pm = J_1 \pm \mathrm{i}J_2$ and $w = x + \mathrm{i}y$, $\bar{w} = x - \mathrm{i}y$.) In terms of horospherical coordinates u and v this means that the potential can be represented in the form

$$V = f(v) + v^2 h(u)$$

for some functions f and h. We now assume that there is a second constant of the motion. It can be taken in the form

$$A_2 = A{J_+}^2 + BJ_3J_- + CJ_3J_+ + DJ_3^2 + d_2(u, v). \tag{21}$$

One can show that the case $A \neq 0$ does not admit any non-constant potentials. Similarly, the case $A = 0, C \neq 0$ does not occur. If $A = C = 0$ and $D \neq 0$, then via a symmetry transformation $\exp(aJ_-)$, for suitable a we leave A_1 unchanged and map A_2 to

$$\tilde{A}_2 = DJ_3^2 + \tilde{d}_2.$$

Thus there are only two cases: (1) $A = C = D = 0$, $B = 1$ and (2) $A = B = C = 0$, $D = 1$.

(S1) $V = \frac{\alpha}{\bar{w}^2} + \frac{\beta z}{\bar{w}^3} + \frac{\gamma(1-4z^2)}{\bar{w}^4}$.
The extra constant has the form [7]

$$A_2 = J_3 J_- + d_2(x, y, z).$$

(S2) $V = \frac{\alpha}{z^2} + \frac{\beta}{\bar{w}^2} + \frac{\gamma w}{\bar{w}^3}$.
The extra constant has the form [7]

$$A_2 = J_3^2 + d_2(x, y, z).$$

There is a special case of **(S2)** that admits an extra symmetry:

(S3) $V = \frac{\alpha}{z^2}$.
The two extra constants are of the form

$$A_2 = (J_1 + \mathrm{i}J_2)^2 + d_2(x, y, z) \qquad A_3 = J_3.$$

For convenience, we will adopt a modified basis given by

$$\tilde{A}_1 = J_1^2 + \frac{\alpha(1 + y^2 - x^2)}{2z^2} \qquad \tilde{A}_2 = J_1 J_2 - \alpha\frac{xy}{z^2} \qquad X = J_3.$$

The Poisson bracket relations are

$$\{X, \tilde{A}_1\} = -2\tilde{A}_2 \qquad \{X, \tilde{A}_2\} = -A_0 + X^2 + 2\tilde{A}_1$$
$$\{\tilde{A}_1, \tilde{A}_2\} = -X(2\tilde{A}_1 + \alpha)$$

with functional relation

$$\tilde{A}_1(A_0 - \tilde{A}_1 - X^2) - \tilde{A}_2^2 - \frac{\alpha}{2}(X^2 + A_0) + \frac{\alpha^2}{4} = 0.$$

We now consider degenerate elliptical coordinates of type 2. The defining constant of the motion has the form

$$A_1 = J_3 J_- + d_1(x, y, z).$$

There is only one (non-degenerate) new system [7]:

(S4) $V = \frac{\alpha}{\bar{w}^2} + \frac{\beta z}{\sqrt{x^2+y^2}} + \frac{\gamma}{\bar{w}\sqrt{x^2+y^2}}$
with constant of the motion

$$A_2 = J_3^2 + d_2(x, y, z).$$

There are two special cases of **(S4)** that admit an extra constant of the motion:

(S5) $V = \frac{\alpha}{\bar{w}^2}$
where the extra constants can be taken as

$$A_1 = J_3 J_- - \frac{\alpha z}{\bar{w}} \qquad A_2 = J_3^2 + \alpha\frac{w}{\bar{w}} \qquad X = J_-.$$

The Poisson bracket relations take the form

$$\{X, A_1\} = \mathrm{i}X^2 - \mathrm{i}\alpha \qquad \{X, A_2\} = 2\mathrm{i}A_1 \qquad \{A_1, A_2\} = 2\mathrm{i}XA_2$$

with the functional relation

$$A_1^2 - A_2 X^2 + \alpha(A_2 - A_0) = 0.$$

(S6) $V = \frac{\alpha z}{\sqrt{x^2+y^2}}$.

A suitable choice of basis is given by

$$A_2 = J_1 J_3 - \frac{\alpha}{2}\frac{x}{\sqrt{x^2+y^2}} \qquad A_3 = J_2 J_3 - \frac{\alpha}{2}\frac{y}{\sqrt{x^2+y^2}} \qquad X = J_3.$$

The Poisson bracket relations are

$$\{X, A_2\} = -A_3 \qquad \{X, A_3\} = A_2 \qquad \{A_2, A_3\} = X(A_0 - 2X^2)$$

with the functional relation

$$A_2^2 + A_3^2 + X^4 - A_0 X^2 - \frac{\alpha^2}{4} = 0.$$

For degenerate elliptical coordinates of type 1 the constant describing this system has the form

$$A_1 = {J_+}^2 - J_3^2 + d_1(x, y, z). \tag{22}$$

Two new (non-degenerate) potentials arise:

(S7) $V = \frac{\alpha x}{\sqrt{y^2+z^2}} + \frac{\beta y}{z^2\sqrt{y^2+z^2}} + \frac{\gamma}{z^2}$.

The extra constant has the form

$$A_2 = J_1^2 + d_2(x, y, z)$$

(see [7]).

(S8) $V = \frac{\alpha x}{\sqrt{y^2+z^2}} + \frac{\beta(w-z)}{\sqrt{w(z-\mathrm{i}y)}} + \frac{\gamma(w+z)}{\sqrt{w(z+\mathrm{i}y)}}$

with the second constant given by

$$A_2 = J_3 J_1 + d_2(x, y, z).$$

There are no special potentials that give a third constant of the motion in this case.

We now consider spherical coordinates. Here, the first constant has the form

$$A_1 = J_3^2 + d_1(x, y, z).$$

There is one (non-degenerate) new system [7]. The potential is:

(S9) $V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}$.

The extra constant has the form

$$A_2 = J_2^2 + d_2(x, y, z).$$

All of the elliptical coordinate cases have already been covered in the cases above. This completes the list of possible superintegrable potentials on the complex two-sphere.

4. Conclusions

In this paper we have, in complete generality, enumerated all potentials on two-dimensional complex constant-curvature spaces for which there is more than one constant of the motion that is quadratic in the momenta. For each pair of constants of the motion, whose leading terms are second order in the enveloping algebra of the Lie symmetry algebra of the free-particle Hamiltonian, we find a pair of coupled second-order linear partial differential equations satisfied by the potential function. The key to making our approach practical is that when one of the constants of the motion corresponds to a separable coordinate system we can explicitly (and simply) solve one of these PDEs in this coordinate system, and merely have to substitute the solution into the second equation.

One can see by inspection of tables A.1 and B.2 that each of these cases (except one) is multiseparable, i.e. separation is possible in at least two coordinate systems. The one counterexample in flat space **(E15)** still separates in one system. These tables also show that each potential listed can be uniquely identified by its list of associated equivalence classes of quadratic constants. This serves to confirm that they are indeed distinct potentials, unrelated by group motions.

We also observe that whenever there is more than one extra quadratic constant a first-order constant can be found. Further, the non-degenerate potentials found in [6,7] that are not related to a degenerate potential by group motions are those for which the additional constants are genuinely second order, i.e. no first-order constant exists.

Note further that for a non-degenerate potential in flat space we can prescribe V, V_x, V_y and V_{yy} arbitrarily at any regular point (x_0, y_0) and these values determine $V(x, y)$ uniquely. These potentials correspond to exactly three functionally and linearly independent constants of the motion. For a degenerate potential with an extra (linearly independent) constant of the motion the additional constant implies a relationship between V_x and V_y at any regular point and hence that all first-, second- and higher-order derivatives of $V(x, y)$ can be expressed in terms of a single first derivative, say V_x. Thus for all these potentials we can prescribe V and V_x arbitrarily at any regular point and these values determine $V(x, y)$ uniquely. It follows that except for the exceptional case **(E15)** the superintegrable potentials depend on exactly four or two parameters. Analogous comments hold for the complex two-sphere, except that here there is no exceptional case.

What is exceptional about **(E15)**? This is the only case where one cannot solve for $V_{xx}-V_{yy}$ and V_{xy} as linear combinations of V_x, V_y. Thus the potential must be degenerate. Indeed this potential, although it depends on an infinite number of parameters, must have the form $V(\bar{z})$ so V_x and V_y cannot be prescribed independently at a point. Furthermore, the potential is not uniquely determined by the values of V, V_x, V_y and V_{yy} at a point.

We give in this paper, and preceding papers, the structure of the classical quadratic algebras in almost all cases. We intend to perform a comprehensive study of the corresponding quantum algebras associated with the Schrödinger equation at a later date.

Appendix A. Separable coordinates in $E_{2,C}$

Each coordinate system in which the Hamilton–Jacobi equation is separable on $E_{2,C}$ is characterized by a constant quadratic in the momenta. Coordinate systems that are related by Euclidean group motions belong to the same family and hence a given family of coordinates (e.g. polar coordinates) is associated with an equivalence class of quadratic elements in the enveloping algebra of $e(2, C)$. Two elements are equivalent if one can be transformed into to other by a combination of scalar multiplication, addition of multiples of $p_x^2 + p_y^2$ and Euclidean motions (including reflections). One equivalence class (listed below) is not associated with a separating coordinate system. The following can be taken as a representative list of coordinate systems and corresponding constants.

(a) Cartesian coordinates:

$$x\,,\, y \qquad L = p_x^2.$$

(b) Light cone coordinates:

$$z = x + \mathrm{i}y \qquad \bar{z} = x - \mathrm{i}y \qquad L = (p_x + \mathrm{i}p_y)^2.$$

(c) Polar coordinates:

$$x_S = r\cos\theta \qquad y_S = r\sin\theta \qquad L = M^2.$$

(d) Semi-hyperbolic coordinates.

$$x_{\rm SH} = {\rm i}(w-u)^2 + 2{\rm i}(w+u) \qquad y_{\rm SH} = -(w-u)^2 + 2(w+u)$$

$$L = M(p_x + {\rm i}p_y) + (p_x - {\rm i}p_y)^2.$$

(e) Hyperbolic coordinates.

$$x_{\rm H} = \frac{r^2+s^2+r^2s^2}{2rs} \qquad y_{\rm H} = {\rm i}\frac{r^2+s^2-r^2s^2}{2rs} \qquad L = M^2 + (p_x + {\rm i}p_y)^2.$$

(f) Parabolic coordinates.

$$x_{\rm P} = \xi\eta \qquad y_{\rm P} = \tfrac{1}{2}(\xi^2 - \eta^2) \qquad L = Mp_x.$$

(g) Elliptic coordinates.

$$x_{\rm E} = c\sqrt{(u-1)(v-1)} \qquad y_{\rm E} = c\sqrt{-uv} \qquad L = M^2 + c^2p_x^2.$$

(h) No separation.

$$\text{No corresponding separable coordinates} \qquad L = M(p_x + {\rm i}p_y).$$

The following facts are useful in determining to which class given constant belongs.

- Translations leave p_x and p_y unchanged and for any A and B a translation can be found that has the effect

$$M \to M + Ap_x + Bp_y.$$

- Rotations leave M fixed and one can be found that for any A and B has the effect

$$Ap_x + Bp_y \to \text{one of } p_x,\ p_+ \text{ or } p_-.$$

- A rotation can be found that for any $A \neq 0$ has the effect

$$p_+ \to Ap_+ \qquad \text{and} \qquad p_- \to \frac{1}{A}p_-.$$

- The reflection $y \to -y$ has the effect

$$M \to -M \qquad p_+ \longleftrightarrow p_-.$$

For each superintegrable potential **(E1–20)**, all linear combinations of the given quadratic constants must be considered in order to determine which equivalence classes are represented, and hence in which families of coordinates systems it will separate.

For example, the potential **(E18)** has constants with leading part $L = AM^2 + BMp_x + CMp_y + D(p_x^2 + p_y^2)$. From this we can see immediately that polar and parabolic coordinates will separate this Hamiltonian, and that the non-separating constant Mp_+ can be generated. Further, $M^2 + 2Mp_y + 2(p_x^2 + p_y^2) \to M^2 + p_x^2$ under a translation that maps $M \to M - p_y$ and hence **(E18)** separates in an elliptic coordinate system. Lastly, there exists a translation mapping $M^2 + 2{\rm i}Mp_+ \to M^2 + p_+^2$, and hence the Hamiltonian separates in hyperbolic coordinates.

The results of similar reasoning for the other potentials are summarized in table A.1.

Table A.1. Separating coordinate systems for superintegrable potentials in complex two-dimensional Euclidean space. Potentials possessing a quadratic constant equivalent to Mp_+ are indicated in the line labelled 'non-separating'.

	E1	**E2**	**E3**	**E4**	**E5**	**E6**	**E7**	**E8**	**E9**	**E10**
Cartesian	×	×	×	×	×	×				
Light cone			×	×	×		×	×	×	×
Polar	×		×			×		×		
Semi-hyperbolic				×						×
Hyperbolic			×				×	×		
Parabolic		×			×	×			×	
Elliptic	×		×			×	×			
Non-separating				×						
	E11	**E12**	**E13**	**E14**	**E15**	**E16**	**E17**	**E18**	**E19**	**E20**
Cartesian										
Light cone	×	×	×	×	×					
Polar				×		×	×	×		
Semi-hyperbolic	×		×							
Hyperbolic		×		×			×	×	×	
Parabolic			×			×		×		××
Elliptic		×				×		×	×	
Non-separating	×	×	×	×	×		×	×	×	×

Appendix B. Separable coordinates in $S_{2,C}$

As for Euclidean space, coordinates separating the Hamilton–Jacobi equation on the two-sphere correspond to constants that are quadratic in the elements of the Lie algebra of its symmetry group O(3). Coordinates belong to the same family if one can be transformed to the other by a rotation or reflection. On the complex two-sphere, unlike complex Euclidean space, every quadratic constant, other than a multiple of the Hamiltonian, corresponds to a separating coordinate system.

The separable coordinates on the complex two-sphere and their characterizing constants areas follows.

(a) Spherical coordinates:

$$x = \sin\theta\cos\varphi \qquad y = \sin\theta\sin\varphi$$

$$z = \cos\theta \qquad L = J_3^2.$$

(b) Horospherical coordinates:

$$x = \frac{\mathrm{i}}{2}\left(v + \frac{u^2-1}{v}\right) \qquad y = \frac{1}{2}\left(v + \frac{u^2+1}{v}\right)$$

$$z = \frac{\mathrm{i}u}{v} \qquad L = (J_1 - \mathrm{i}J_2)^2.$$

(c) Elliptic coordinates:

$$x^2 = \frac{(ru-1)(rv-1)}{1-r} \qquad y^2 = \frac{r(u-1)(v-1)}{r-1}$$

$$z^2 = ruv \qquad L = J_1^2 + rJ_2^2.$$

Table B.1. Invariants used to identify coordinate systems on $S_{2,C}$.

Coordinate system	No of distinct eigenvalues	dim ker(ϕ)
Spherical	2	1
Horospherical	1	1
Elliptic	3	0
Degenerate elliptic type 1	2	0
Degenerate elliptic type 2	1	0

Table B.2. Separating coordinate systems for superintegrable potentials on the two-dimensional complex sphere.

	S1	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**	**S9**
Spherical		×	×	×	×	×	×		×
Horospherical	×	×	×		×				
Elliptic	×		×		×	×	×	×	×
Degenerate elliptic 1		×	×	×	×	×	×	×	
Degenerate elliptic 2	×			×	×	×			

(d) Degenerate elliptic coordinates of type 1:

$$x + \mathrm{i}y = \frac{4uv}{(u^2+1)(v^2+1)} \qquad x - \mathrm{i}y = \frac{(u^2v^2+1)(u^2+v^2)}{uv(u^2+1)(v^2+1)}$$

$$z = \frac{(u^2-1)(v^2-1)}{(u^2+1)(v^2+1)} \qquad L = (J_1 + \mathrm{i}J_2)^2 - J_3^2.$$

(e) Degenerate elliptic coordinates of type 2:

$$x + \mathrm{i}y = -\mathrm{i}uv \qquad x - \mathrm{i}y = \frac{1}{4}\frac{(u^2+v^2)^2}{u^3v^3}$$

$$z = \frac{\mathrm{i}}{2}\frac{u^2-v^2}{uv} \qquad L = J_3(J_1 - \mathrm{i}J_2).$$

The action of the symmetry group on a general quadratic constant is not as easily described as for $E_{2,C}$. To determine the equivalence class to which a given quadratic element L belongs it is more convenient to note that the number of distinct eigenvalues of L, as a quadratic form in the J_i, and the dimension of the kernel of the map on first-order elements, $\phi : X \mapsto \{X, L\}$, are both invariant under group motions and addition of multiples of the Casimir $J_1^2 + J_2^2 + J_3^2$. Table B.1 gives the correspondence between these invariants and families of coordinate systems on $S_{2,C}$.

Just as for $E_{2,C}$, by considering a general linear combination of constants for each potential **(S1–S9)**, the corresponding families of separable coordinates can be determined. The results are summarized in table B.2.

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